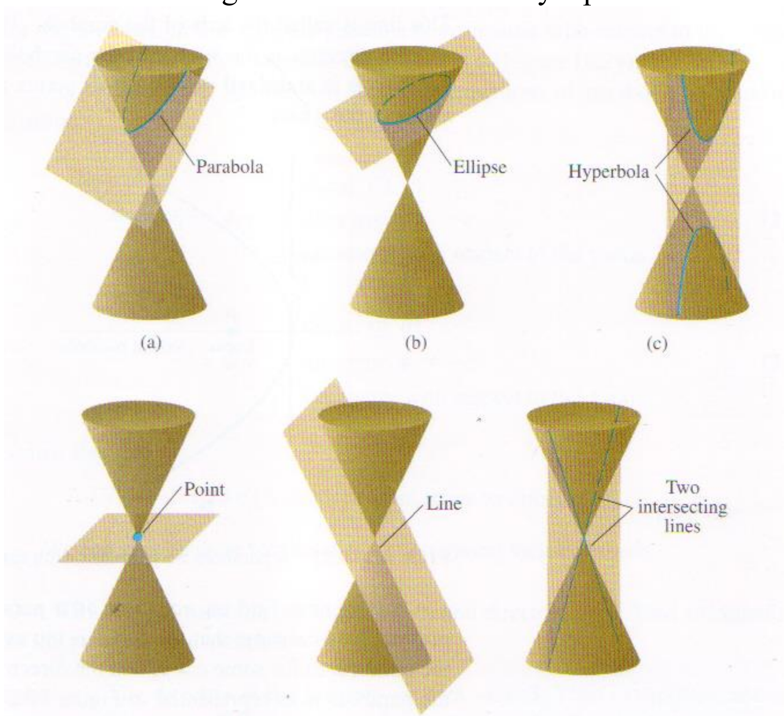
Calculus 140, section 10.3 Conic Sections

notes by Tim Pilachowski

"The conic sections arise when a double right circular cone is cut by a plane."

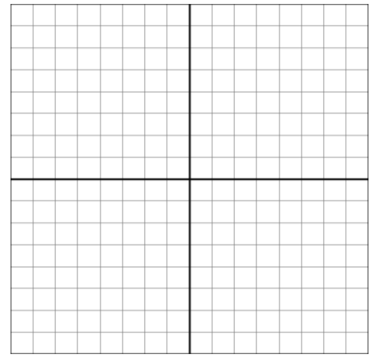


"Any second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is (except in degenerate cases) an equation of a parabola, an ellipse, or a hyperbola." By using completing the square, we can determine shifts/translations $(x-h)$ and $(y-k)$ from "standard" position. (You'll need this for some homework exercises.)

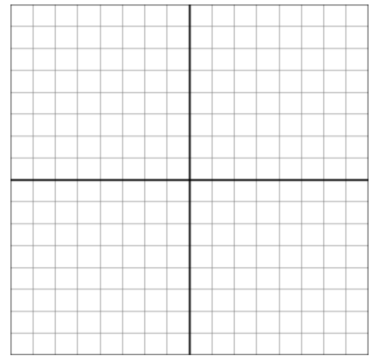
In the *xy* real number plane, a degenerate conic can be one of the three cases shown in the lower pictures above, or the null set (i.e., no points).

conic section	basic equation	reference line	focus/ foci	vertex/ vertices	standard position ($h = k = 0$) symmetry	asymptote(s)
parabola						
ellipse						
hyperbola						

You know the basic quadratic function from Algebra, $y = x^2$, and its shape: a parabola.



If we turn that parabolic shape clockwise by 90 degrees, we get a slightly different equation: $x = y^2$.



Definition 10.1: "Let *l* be a fixed line and *P* a fixed point not on *l*. The set of all points in the plane equidistant from *l* and *P* is called a **parabola**."

The line *l* is called the **directrix** and the point *P* is called the **focus**.

Applications:

The point midway between directrix and focus is the **vertex**. (You used this designation in Algebra.)

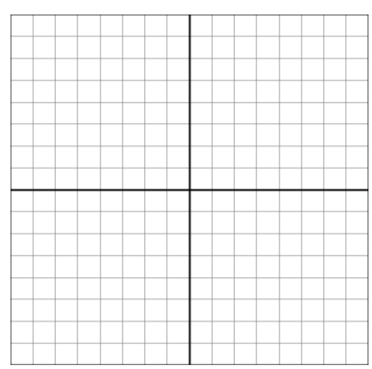
"Standard position" is with the *x*-axis or the *y*-axis as the axis of symmetry.

The text uses the distance formula to derive standard forms $x^2 = 4cy$ and $y^2 = 4cx$.
A parabola has no asymptotes.

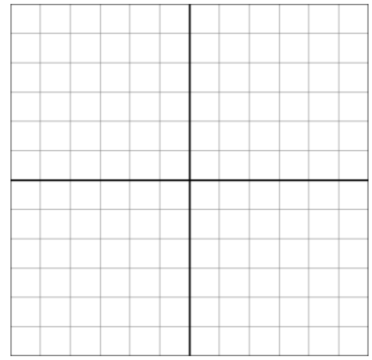
Other applications:

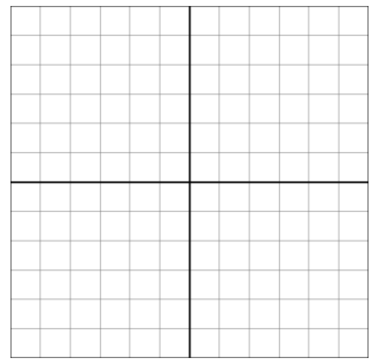
The non-standard position forms are $(x-h)^2 = 4c(y-k)$ and $(y-k)^2 = 4c(x-h)$.
We can use shifts/translations (just like in Algebra) to help us graph or answer questions.
For finding derivatives, we have implicit differentiation (section 3.6).

Example A. The vertex is (–2, 0), and the directrix is $x = \frac{3}{2}$. Identify the focus of the parabola, find its equation, and then sketch the graph.



Definition 10.2: "Let P_1 and P_2 be two points in the plane, and let *k* be a number greater than the distance between P_1 and P_2. The set of all points *P* in the plane such that $|P_1P| + |P_2P| = k$ is called an **ellipse**."





The text uses the distance formula to derive standard forms $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $0 < b \le a$.

Note that a larger denominator under x^2 gives a horizontal orientation to an ellipse, and a larger denominator under y^2 gives a vertical orientation to an ellipse.

The reference lines are called the **major axis** and the **minor axis**, with the major axis being the longer of the two
(or in the special case of a circle, equal in length).
Geometric interpretation: 2*a* is the length of the major axis, and 2*b* is the length of the minor axis.

In standard position, the major axis will have equation either *y* = 0 (horizontal orientation) or *x* = 0 (vertical orientation).

The points P_1 and P_2 are called the **foci** (the plural of focus).

Given $c = \sqrt{a^2 - b^2}$ (Pythagorean/distance formula), in standard position, the foci will have coordinates of

either (–*c*, 0) and (*c*, 0) when there is a horizontal orientation

or (0, –*c*) and (0, *c*) when there is a vertical orientation.

Geometric interpretation: 2*c* is the distance between the two foci.

When *a* = *b*, the major axis and the minor axis have the same length, and the two foci are a single point: the center of a circle.

Applications:

The points where the major axis intersects the ellipse are the **vertices**.

In standard position, the vertices will have coordinates of

either (–*a*, 0) and (*a*, 0) when there is a horizontal orientation

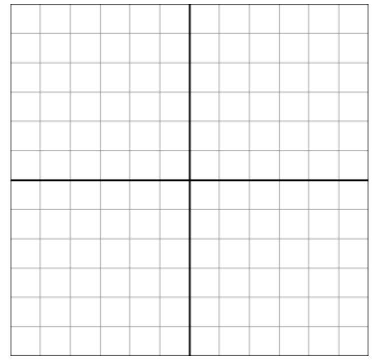
or (0, –*a*) and (0, *a*) when there is a vertical orientation.

In standard position, an ellipse (either orientation) will be symmetric with respect to the *x*-axis, the *y*-axis and also the origin.

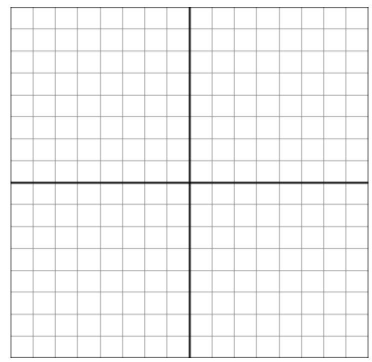
An ellipse has no asymptotes.

The non-standard position forms are $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ and $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$.
We can use shifts/translations (just like in Algebra) to help us graph or answer questions.
For finding derivatives, we have implicit differentiation (section 3.6).

Example B. An ellipse passes through the points $\left(1, \frac{\sqrt{27}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{135}}{4}\right)$ and is in standard position. Find the equation and sketch the graph.



Definition 10.3: "Let P_1 and P_2 be two points in the plane, and let k be a positive number less than the distance between P_1 and P_2. The set of all points P in the plane such that $\left|\,|P_1P| - |P_2P|\,\right| = k$ called a **hyperbola**."



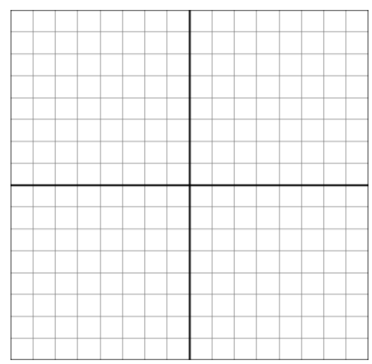
The text uses the distance formula to derive standard forms $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

The points P_1 and P_2 are called the **foci** (the plural of focus).

Given $c = \sqrt{a^2 + b^2}$ (Pythagorean/distance formula), in standard position, the foci will have coordinates of

either $(-c, 0)$ and $(c, 0)$ when there is a horizontal orientation

or $(0, -c)$ and $(0, c)$ when there is a vertical orientation.



Geometric interpretation: $2c$ is the distance between the two foci.

The point located halfway between the two foci is called the **center** of the hyperbola. In standard position, the center of the hyperbola will have coordinates (0, 0).

The reference line, the line through the two foci, is called the **principal axis**.

In standard position, the major axis will have equation either $y = 0$ (horizontal orientation) or $x = 0$ (vertical orientation).

The points where the principal axis intersects the hyperbola are the **vertices**.

In standard position, the vertices will have coordinates of

either $(-a, 0)$ and $(a, 0)$ when there is a horizontal orientation

or $(0, -a)$ and $(0, a)$ when there is a vertical orientation.

The line segment connecting the two vertices is called the **transverse axis** of the hyperbola.

Geometric interpretation: $2a$ is the length of the transverse axis.

In standard position, a hyperbola (either orientation) will be symmetric with respect to the x-axis, the y-axis and also the origin.

A hyperbola has two asymptotes: $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ (when there is a horizontal orientation) or $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$ (when there is a vertical orientation).

Examples of hyperbolic applications are noted in the text.

The non-standard position forms are $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.
We can use shifts/translations (just like in Algebra) to help us graph or answer questions.
For finding derivatives, we have implicit differentiation (section 3.6).

Example C. A hyperbola has foci $\left(0, -\sqrt{13}\right)$ and $\left(0, \sqrt{13}\right)$, has asymptotes $y = -\frac{2}{3}x$ and $y = \frac{2}{3}x$, and is in standard position. Find the equation and sketch the graph.